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LINEARIZED EQUATIONS OF NONLINEAR ELASTIC DEFORMATION OF THIN PLATES

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A linearized system of equations governing elastic deformation of a thin plate with arbitrary boundary conditions at its faces in an arbitrary curvilinear coordinate system is proposed. This system of equations is the first approximation of a one-parameter sequence of equations of two-dimensional problems obtained from the initial three-dimensional problem by approximating unknown functions by truncated series in Legendre polynomials. The stability problem of an infinite plate compressed uniaxially is solved. The results obtained are compared with the existing solutions.

The existing procedures of constructing equations governing elastic deformation of plates can be arbitrarily divided into two groups. The first group consists of methods based on simplifying hypotheses (classical theory, equations of the Timoshenko type, etc.). The second group comprises methods that reduce the initial three-dimensional problem to a sequence of two-dimensional problems (asymptotic methods and expansion in thickness with the use of various basis functions). As the basis functions, the Legendre polynomials are usually used (see, e.g., [1]). Ivanov [2] proposed a method of constructing equations of elastic deformation of plates and shells of constant thickness with arbitrary boundary conditions for displacements and stresses at the surfaces. This method is based on several approximations of the same unknown functions by truncated series in Legendre polynomials. Using this method, Alekseev [3] obtained a one-parameter family of successive approximations of the equations of a deformable layer of variable thickness in arbitrary curvilinear coordinates. Alekseev [4] generalized the method proposed in [2, 3] to nonlinear elastic deformation of plates.

In the present paper, the system of equations of the first approximation of the sequence of the equations obtained in [4] is linearized.

1. Equations of the Nonlinear Theory of Elasticity in Arbitrary Curvilinear Coordinates. We consider an arbitrary curvilinear system of Lagrangian coordinates ξ^i $(i = 1, 2, 3)$. The equations of equilibrium of a continuous medium are written in the vector form as

$$\hat{\boldsymbol{t}}^i_{,i} + \hat{\boldsymbol{f}} = 0, \qquad \hat{\boldsymbol{t}}^i = J\boldsymbol{t}^i, \qquad \hat{\boldsymbol{f}} = J\boldsymbol{f}, \qquad \boldsymbol{t}^i = \sigma^{ij}\boldsymbol{g}_j, \tag{1.1}$$

where $\boldsymbol{g}_i$ is the covariant basis of the curvilinear coordinate system ξ^i in a deformed state, $J = \boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3)$ is the Jacobian of transformation of the coordinates, σ^{ij} are the components of the Cauchy stress tensor, and $\boldsymbol{f}$ is the vector of body forces.

The components of the Green–Lagrange strain tensor ε_{ij} are related to the displacement vector $\boldsymbol{u}$ by the nonlinear relations

$$2\varepsilon_{ij} = \overset{0}{\boldsymbol{g}}_i \cdot \boldsymbol{u}_{,j} + \overset{0}{\boldsymbol{g}}_j \cdot \boldsymbol{u}_{,i} + \boldsymbol{u}_{,i} \cdot \boldsymbol{u}_{,j}, \tag{1.2}$$

where $\overset{0}{\boldsymbol{g}}_i$ is the covariant basis of the coordinate system ξ^i in the undeformed state; the superimposed zero shows that the quantity corresponds to the undeformed state.

The covariant basis of the coordinate system ξ^i in the deformed state is

$$\boldsymbol{g}_i = \overset{0}{\boldsymbol{g}}_i + \boldsymbol{u}_{,i}. \tag{1.3}$$

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Hooke's law is taken in the form

$$\tau^{ij} = \overset{0}{C}{}^{ijks}\varepsilon_{ks}, \tag{1.4}$$

where τ^{ij} are the contravariant components of the second Piola–Kirchhoff stress tensor, and $\overset{0}{C}{}^{ijks}$ are the contravariant components of the fourth-rank tensor, which satisfy the symmetry conditions $\overset{0}{C}{}^{ijks} = \overset{0}{C}{}^{jiks} = \overset{0}{C}{}^{ksij}$.

In the coordinate system ξ^i, the following equality is valid:

$$\overset{0}{J}\tau^{ij} = J\sigma^{ij}. \tag{1.5}$$

Here $\overset{0}{J} = \overset{0}{\boldsymbol{g}}_1 \cdot (\overset{0}{\boldsymbol{g}}_2 \times \overset{0}{\boldsymbol{g}}_3)$ is the Jacobian of coordinate transformation in the initial-space metric.

Below, the boundary conditions refer to the undeformed state.

We assume that the boundary $\overset{0}{S}$ of an undeformed body consists of two parts: part $\overset{0}{S}_u$, where the displacements

$$\boldsymbol{u}\Big|_{\overset{0}{S}_u} = \boldsymbol{u}_* \tag{1.6}$$

are specified, and part $\overset{0}{S}_\sigma$, where the stresses

$$\tau^{ij}\boldsymbol{g}_j\overset{0}{\nu}_i\Big|_{\overset{0}{S}_\sigma} = \boldsymbol{p}_* \tag{1.7}$$

are specified. Here $\overset{0}{\nu}_i$ are the direction cosines of the outward normal vector to the boundary $\overset{0}{S}$ and $\boldsymbol{u}_*$ and $\boldsymbol{p}_*$ are the vector functions specified on $\overset{0}{S}$.

Relative to the undeformed state, the boundary-value problem (1.1)–(1.7) is taken to be the initial boundary-value problem of the nonlinear theory of elasticity.

2. Linearized Equations of the First Approximation in the Case where the Current State is Described by Geometrically Nonlinear Equations. We consider a plate of constant thickness $2h$. In the undeformed state, the plate occupies the volume $\overset{0}{V}$ bounded by the faces $\overset{0}{S}{}^+$ and $\overset{0}{S}{}^-$ and edge surface $\overset{0}{\Sigma}$.

Let x_i be the Cartesian coordinates. In the undeformed state, the middle surface of the plate coincides with the coordinate plane $x_3 = 0$, and the faces $\overset{0}{S}{}^+$ and $\overset{0}{S}{}^-$ correspond to $x_3 = +h$ and $x_3 = -h$, respectively.

We choose the curvilinear system of Lagrangian coordinates ξ^k in such a manner that the ξ^3 axis coincides with the x_3 axis in the undeformed state. The coordinates x_3 and ξ^3 are related by the formula $x_3 = h\xi^3$. In the undeformed state, the position of any internal point in the plate of volume $\overset{0}{V}$ is determined by the vector function of the curvilinear coordinates ξ^k:

$$\overset{0}{\boldsymbol{R}}(\xi^k) = \overset{0}{\boldsymbol{r}}(\xi^\alpha) + h\boldsymbol{n}\xi^3 \qquad (\xi^k \in V_\xi \subset \mathbb{R}^3). \tag{2.1}$$

Here $V_\xi = \{\xi^k \mid \xi^\alpha \in S_\xi \subset \mathbb{R}^2,\ \xi^3 \in [-1,1]\}$ and $\boldsymbol{n}$ is the unit vector directed along the x_3 axis.

It follows from (2.1) that the covariant local basis of the coordinate system ξ^k in the undeformed state has the form

$$\overset{0}{\boldsymbol{g}}_\alpha = \overset{0}{\boldsymbol{R}}_{,\alpha} = \overset{0}{\boldsymbol{r}}_{,\alpha}, \qquad \overset{0}{\boldsymbol{g}}_3 = \overset{0}{\boldsymbol{R}}_{,3} = h\boldsymbol{n}. \tag{2.2}$$

One can see from relations (2.2) that the vectors $\overset{0}{\boldsymbol{g}}_\alpha$ depend only on the coordinates ξ^α, whereas the vector $\overset{0}{\boldsymbol{g}}_3$ is independent of ξ^k.

Since $\xi^3 \in [-1,1]$, the unknown functions $\boldsymbol{u}$ and $\hat{\boldsymbol{t}}^i$ can be expanded in series in terms of Legendre polynomials:

$$\boldsymbol{u} = \sum_{k=0}^{\infty}[\boldsymbol{u}]^k P_k, \qquad \hat{\boldsymbol{t}}^i = \sum_{k=0}^{\infty}[\hat{\boldsymbol{t}}^i]^k P_k.$$

Here $P_k(\xi^3)$ are the orthogonal Legendre polynomials and $[\boldsymbol{u}]^k$ and $[\hat{\boldsymbol{t}}^i]^k$ are the expansion coefficients which depend on the coordinates $\{\xi^\alpha\} \in S_\xi \subset \mathbb{R}^2$:

$$[\boldsymbol{u}]^k = \frac{1+2k}{2}\int\limits_{-1}^{1} \boldsymbol{u} P_k\, d\xi^3, \qquad [\hat{\boldsymbol{t}}^i]^k = \frac{1+2k}{2}\int\limits_{-1}^{1} \hat{\boldsymbol{t}}^i P_k\, d\xi^3.$$

Alekseev [4] obtained an one-parameter family of successive Nth approximations of the nonlinear equations of elastic deformation of plates. The first approximation ($N = 0$) is of greatest interest. In this case, we introduce the following two approximations of the displacement vector $\boldsymbol{u}$:

$$\boldsymbol{u} \sim \boldsymbol{U}' = \overset{0}{\boldsymbol{g}}{}^{\alpha}\Big(\overset{0}{\boldsymbol{g}}_{\alpha} \cdot \sum_{k=0}^{1} [\boldsymbol{u}]^k P_k\Big) + \overset{0}{\boldsymbol{g}}{}^{3}(\overset{0}{\boldsymbol{g}}_{3} \cdot [\boldsymbol{u}]^0),$$
$$\boldsymbol{u} \sim \boldsymbol{U}'' = \overset{0}{\boldsymbol{g}}{}^{\alpha}\Big(\overset{0}{\boldsymbol{g}}_{\alpha} \cdot \sum_{k=0}^{3} [\boldsymbol{u}]^k P_k\Big) + \overset{0}{\boldsymbol{g}}{}^{3}\Big(\overset{0}{\boldsymbol{g}}_{3} \cdot \sum_{k=0}^{2} [\boldsymbol{u}]^k P_k\Big). \tag{2.3}$$

In accordance with (2.3), we use the following approximations of the covariant basis of the deformed state $\boldsymbol{g}_i$ in (1.3):

$$\boldsymbol{g}_{\alpha} \sim \boldsymbol{G}_{\alpha} = \overset{0}{\boldsymbol{g}}_{\alpha} + \boldsymbol{U}'_{,\alpha}, \qquad \boldsymbol{g}_3 \sim \boldsymbol{G}_3 = \overset{0}{\boldsymbol{g}}_3 + \boldsymbol{U}''_{,3}. \tag{2.4}$$

In (2.4), the vectors $\boldsymbol{U}'$ and $\boldsymbol{U}''$ are approximations of the displacement vector $\boldsymbol{u}$, which differ in the number of terms in the series. They are used to calculate the derivative with respect to the coordinates ξ^{α} and the coordinate ξ^3, respectively.

The equations of equilibrium (1.1) are approximated by the relations

$$\overset{0}{\boldsymbol{g}}{}^{\alpha} \cdot (\hat{\boldsymbol{T}}'^{i}_{,i} + \hat{\boldsymbol{F}}) = 0, \qquad \overset{0}{\boldsymbol{g}}{}^{3} \cdot (\hat{\boldsymbol{T}}''^{i}_{,i} + \hat{\boldsymbol{F}}) = 0. \tag{2.5}$$

Here, the quantities $\hat{\boldsymbol{T}}'^{i}$, $\hat{\boldsymbol{T}}''^{i}$, and $\hat{\boldsymbol{F}}$ correspond to the truncated series

$$\hat{\boldsymbol{T}}'^{\alpha} = \sum_{k=0}^{1} [\hat{\boldsymbol{t}}^{\alpha}]^k P_k, \quad \hat{\boldsymbol{T}}''^{\alpha} = [\hat{\boldsymbol{t}}^{\alpha}]^0, \quad \hat{\boldsymbol{T}}'^{3} = \hat{\boldsymbol{T}}''^{3} = \overset{0}{\boldsymbol{g}}_{\alpha} \sum_{k=0}^{2} ([\hat{\boldsymbol{t}}^{3}]^k \cdot \overset{0}{\boldsymbol{g}}{}^{\alpha}) P_k + \overset{0}{\boldsymbol{g}}_3 \sum_{k=0}^{1} ([\hat{\boldsymbol{t}}^{3}]^k \cdot \overset{0}{\boldsymbol{g}}{}^{3}) P_k,$$

$$\hat{\boldsymbol{F}} = \overset{0}{\boldsymbol{g}}_{\alpha} \sum_{k=0}^{1} ([\,\hat{\boldsymbol{f}}\,]^k \cdot \overset{0}{\boldsymbol{g}}{}^{\alpha}) P_k + \overset{0}{\boldsymbol{g}}_3 ([\hat{\boldsymbol{f}}]^0 \cdot \overset{0}{\boldsymbol{g}}{}^{3}).$$

Thus, the same quantities $\hat{\boldsymbol{t}}^{\alpha}$ in (2.5) are approximated by two expressions: the approximation $\hat{\boldsymbol{T}}'^{\alpha}$ is used in the equations of equilibrium in the coordinate plane ξ^{α}, whereas the approximation $\hat{\boldsymbol{T}}''^{\alpha}$ is used to formulate the condition of equilibrium in the transverse direction.

The components of the Green–Lagrange strain tensor ε_{ij} (1.2) are approximated as

$$\varepsilon_{\alpha\beta} \sim 2E_{\alpha\beta} = \overset{0}{\boldsymbol{g}}_{\beta} \cdot \boldsymbol{U}'_{,\alpha} + \overset{0}{\boldsymbol{g}}_{\alpha} \cdot \boldsymbol{U}'_{,\beta} + \boldsymbol{U}'_{,\alpha} \cdot \boldsymbol{U}'_{,\beta},$$

$$\varepsilon_{3\alpha} \sim 2E_{3\alpha} = \overset{0}{\boldsymbol{g}}_{\alpha} \cdot \boldsymbol{U}''_{,3} + \overset{0}{\boldsymbol{g}}_{3} \cdot \boldsymbol{U}'_{,\alpha} + \boldsymbol{U}'_{,\alpha} \cdot \boldsymbol{U}''_{,3}, \qquad \varepsilon_{33} \sim E_{33} = \overset{0}{\boldsymbol{g}}_{3} \cdot \boldsymbol{U}''_{,3} + 0.5\boldsymbol{U}''_{,3} \cdot \boldsymbol{U}''_{,3}.$$

In addition to the current state, we consider a perturbed state corresponding to perturbed displacements $\tilde{\boldsymbol{U}}'$ and $\tilde{\boldsymbol{U}}''$:

$$\tilde{\boldsymbol{U}}' = \boldsymbol{U}' + \Delta\boldsymbol{U}', \qquad \tilde{\boldsymbol{U}}'' = \boldsymbol{U}'' + \Delta\boldsymbol{U}''.$$

Here, the perturbation vectors $\Delta\boldsymbol{U}'$ and $\Delta\boldsymbol{U}''$ are truncated series similar to series (2.3).

For the vectors of the covariant basis of the perturbed state, we obtain

$$\tilde{\boldsymbol{G}}_i = \boldsymbol{G}_i + \Delta\boldsymbol{G}_i, \qquad \Delta\boldsymbol{G}_{\alpha} = \Delta\boldsymbol{U}'_{,\alpha}, \qquad \Delta\boldsymbol{G}_3 = \Delta\boldsymbol{U}''_{,3}. \tag{2.6}$$

According to [4], the linearized system of equations of the first approximation comprises:

— the equations of equilibrium [approximations of Eqs. (1.1)]

$$\overset{0}{\boldsymbol{g}}{}^{\alpha} \cdot (\Delta\hat{\boldsymbol{T}}'^{i}_{,i} + \Delta\hat{\boldsymbol{F}}) = 0, \qquad \overset{0}{\boldsymbol{g}}{}^{3} \cdot (\Delta\hat{\boldsymbol{T}}''^{i}_{,i} + \Delta\hat{\boldsymbol{F}}) = 0; \tag{2.7}$$

— Hooke's law relations [approximations of Eqs. (1.4) written in the form of truncated series]

$$\Delta\hat{\boldsymbol{T}}'^{\alpha} = \sum_{k=0}^{1} P_k \frac{1+2k}{2} \int_{-1}^{1} \overset{0}{J} \tilde{C}^{\alpha jmn}(\boldsymbol{G}_m \cdot \Delta\boldsymbol{G}_n)\boldsymbol{G}_j P_k \, d\xi^3,$$

$$\Delta\hat{\boldsymbol{T}}''^{\alpha} = \frac{1}{2} \int_{-1}^{1} \overset{0}{J} \tilde{C}^{\alpha jmn}(\boldsymbol{G}_m \cdot \Delta\boldsymbol{G}_n)\boldsymbol{G}_j P_0 \, d\xi^3, \tag{2.8}$$

$$\Delta\hat{\boldsymbol{T}}^{3} = \Delta\hat{\boldsymbol{T}}'^{3} = \Delta\hat{\boldsymbol{T}}''^{3} = \overset{0}{\boldsymbol{g}}_\alpha \sum_{k=0}^{2} \Big(P_k \frac{1+2k}{2} \overset{0}{\boldsymbol{g}}{}^{\alpha} \cdot \int_{-1}^{1} \overset{0}{J} \tilde{C}^{3jmn}(\boldsymbol{G}_m \cdot \Delta\boldsymbol{G}_n)\boldsymbol{G}_j P_k \, d\xi^3 \Big)$$

$$+ \overset{0}{\boldsymbol{g}}_3 \sum_{k=0}^{1} \Big(P_k \frac{1+2k}{2} \overset{0}{\boldsymbol{g}}{}^{3} \cdot \int_{-1}^{1} \overset{0}{J} \tilde{C}^{3jmn}(\boldsymbol{G}_m \cdot \Delta\boldsymbol{G}_n)\boldsymbol{G}_j P_k \, d\xi^3 \Big);$$

— the boundary conditions at the faces [approximations of conditions (1.6) and (1.7)]

$$\Delta\boldsymbol{U}''\Big|_{\overset{0}{S}{}_u^+} = \Delta\boldsymbol{u}_*, \quad \Delta\boldsymbol{U}''\Big|_{\overset{0}{S}{}_u^-} = \Delta\boldsymbol{u}_*, \quad \frac{\Delta\hat{\boldsymbol{T}}^3}{\overset{0}{J}}\Big|_{\overset{0}{S}{}_\sigma^+} = \Delta\boldsymbol{p}_*, \quad \frac{\Delta\hat{\boldsymbol{T}}^3}{\overset{0}{J}}\Big|_{\overset{0}{S}{}_\sigma^-} = \Delta\boldsymbol{p}_*. \tag{2.9}$$

In the equations of equilibrium (2.7), the perturbation vector of the body forces $\Delta\hat{\boldsymbol{F}}$ is approximated by the truncated series

$$\Delta\hat{\boldsymbol{F}} = \overset{0}{\boldsymbol{g}}_\alpha \sum_{k=0}^{1} ([\Delta\hat{\boldsymbol{f}}]^k \cdot \overset{0}{\boldsymbol{g}}{}^{\alpha}) P_k + \overset{0}{\boldsymbol{g}}_3 ([\Delta\hat{\boldsymbol{f}}]^0 \cdot \overset{0}{\boldsymbol{g}}{}^{3}) P_0.$$

In Eqs. (2.8), the components of the fourth-rank tensor $\tilde{C}^{ijmn}$ are given by

$$\tilde{C}^{ijmn} = \overset{0}{C}{}^{ijmn} + \tau^{in} G^{mj}, \qquad G^{mj} = \boldsymbol{G}^m \cdot \boldsymbol{G}^j, \qquad \boldsymbol{G}_j \cdot \boldsymbol{G}^i = \delta^i_j,$$

where δ^i_j is the Kronecker symbol and τ^{ij} are related to the approximations E_{ks} by Hooke's law (1.4). The linear system (2.6)–(2.9) is supplemented by the following linearized boundary conditions at the edge surfaces [approximations of the boundary conditions (1.6) and (1.7)]:

$$\Delta\boldsymbol{U}'\Big|_{\overset{0}{\Sigma}_u} = \Delta\boldsymbol{U}'_*, \qquad \frac{\Delta\hat{\boldsymbol{T}}^{\alpha}\nu^0_\alpha}{J^0}\Big|_{\overset{0}{\Sigma}_\sigma} = \Delta\boldsymbol{P}'_* \qquad (\overset{0}{\Sigma}_u \cup \overset{0}{\Sigma}_\sigma = \overset{0}{\Sigma}),$$

$$\Delta\hat{\boldsymbol{T}}^{\alpha} = \overset{0}{\boldsymbol{g}}_\gamma (\Delta\hat{\boldsymbol{T}}'^{\alpha} \cdot \overset{0}{\boldsymbol{g}}{}^{\gamma}) + \overset{0}{\boldsymbol{g}}_3 (\Delta\hat{\boldsymbol{T}}''^{\alpha} \cdot \overset{0}{\boldsymbol{g}}{}^{3}).$$

Here, the vectors $\Delta\boldsymbol{U}'_*$ and $\Delta\boldsymbol{P}'_*$ are the truncated series

$$\Delta\boldsymbol{U}'_* = \overset{0}{\boldsymbol{g}}{}^{\alpha} (\overset{0}{\boldsymbol{g}}_\alpha \cdot \sum_{k=0}^{1} [\Delta\boldsymbol{u}_*]^k P_k) + \overset{0}{\boldsymbol{g}}{}^{3} (\overset{0}{\boldsymbol{g}}_3 \cdot [\Delta\boldsymbol{u}_*]^0 P_0),$$

$$\Delta\boldsymbol{P}'_* = \overset{0}{\boldsymbol{g}}{}^{\alpha} (\overset{0}{\boldsymbol{g}}_\alpha \cdot \sum_{k=0}^{1} [\Delta\boldsymbol{p}_*]^k P_k) + \overset{0}{\boldsymbol{g}}{}^{3} (\overset{0}{\boldsymbol{g}}_3 \cdot [\Delta\boldsymbol{p}_*]^0 P_0).$$

The linear system (2.7)–(2.9) is a linearized system of nonlinear equations of elastic deformation of thin plates (first approximation) whose differential order is equal to ten [4], and it does not depend on the boundary conditions at the faces (stresses or displacements can be specified).

3. Linearized Equations of the First Approximation in the Case where the Current State is Described by Geometrically Linear Equations. Let

$$\boldsymbol{g}_\alpha \simeq \overset{0}{\boldsymbol{g}}_\alpha. \tag{3.1}$$

Accordingly, in (2.2) we have

$$G_\alpha \simeq \overset{0}{\boldsymbol{g}}_\alpha. \tag{3.2}$$

Substituting (3.1) and (3.2) into Eqs. (2.7)–(2.9), we obtain a linearized system of equations for thin plates, which comprises:

— the equations of equilibrium

$$\overset{0}{\boldsymbol{g}}{}^{\alpha} \cdot (\Delta \hat{\boldsymbol{T}}'^{i}_{,i} + \Delta \hat{\boldsymbol{F}}) = 0, \qquad \overset{0}{\boldsymbol{g}}{}^{3} \cdot (\Delta \hat{\boldsymbol{T}}''^{i}_{,i} + \Delta \hat{\boldsymbol{F}}) = 0; \tag{3.3}$$

— Hooke's law equations

$$\Delta \hat{\boldsymbol{T}}'^{\alpha} = \sum_{k=0}^{1} P_k \, \frac{1+2k}{2} \int_{-1}^{1} \overset{0}{J} \tilde{C}^{\alpha jmn} (\overset{0}{\boldsymbol{g}}_m \cdot \Delta \boldsymbol{G}_n) \overset{0}{\boldsymbol{g}}_j P_k \, d\xi^3,$$

$$\Delta \hat{\boldsymbol{T}}''^{\alpha} = \frac{1}{2} \int_{-1}^{1} \overset{0}{J} \tilde{C}^{\alpha jmn} (\overset{0}{\boldsymbol{g}}_m \cdot \Delta \boldsymbol{G}_n) \overset{0}{\boldsymbol{g}}_j P_0 \, d\xi^3, \tag{3.4}$$

$$\Delta \hat{\boldsymbol{T}}^{3} = \Delta \hat{\boldsymbol{T}}'^{3} = \Delta \hat{\boldsymbol{T}}''^{3} = \overset{0}{\boldsymbol{g}}_{\alpha} \sum_{k=0}^{2} \Big(P_k \, \frac{1+2k}{2} \int_{-1}^{1} \overset{0}{J} \tilde{C}^{3\alpha mn} (\overset{0}{\boldsymbol{g}}_m \cdot \Delta \boldsymbol{G}_n) P_k \, d\xi^3 \Big)$$

$$+ \overset{0}{\boldsymbol{g}}_{3} \sum_{k=0}^{1} \Big(P_k \frac{1+2k}{2} \int_{-1}^{1} \overset{0}{J} \tilde{C}^{33mn} (\overset{0}{\boldsymbol{g}}_m \cdot \Delta \boldsymbol{G}_n) P_k \, d\xi^3 \Big);$$

— the boundary conditions at the faces

$$\Delta \boldsymbol{U}'' \Big|_{\overset{0}{S}{}^{+}_{u}} = \Delta \boldsymbol{u}_*, \quad \Delta \boldsymbol{U}'' \Big|_{\overset{0}{S}{}^{-}_{u}} = \Delta \boldsymbol{u}_*, \quad \frac{\Delta \hat{\boldsymbol{T}}^{3}}{\overset{0}{J}} \Big|_{\overset{0}{S}{}^{+}_{\sigma}} = \Delta \boldsymbol{p}_*, \quad \frac{\Delta \hat{\boldsymbol{T}}^{3}}{\overset{0}{J}} \Big|_{\overset{0}{S}{}^{-}_{\sigma}} = \Delta \boldsymbol{p}_*. \tag{3.5}$$

In Eqs. (3.4), the components of the fourth-rank tensor $\tilde{C}^{ijmn}$ take the form

$$\tilde{C}^{ijmn} = \overset{0}{C}{}^{ijmn} + \tau^{in} \overset{0}{g}{}^{mj}, \qquad \overset{0}{g}{}^{mj} = \overset{0}{\boldsymbol{g}}{}^{m} \cdot \overset{0}{\boldsymbol{g}}{}^{j}. \tag{3.6}$$

As in the previous case, the differential order of the linear system (3.3)–(3.5) is equal to ten.

4. Stability of a Compressed Infinite Plate. We consider an infinitely long plate of width l and thickness $2h$. Body forces are ignored ($\boldsymbol{f} = 0$).

We introduce the coordinate system ξ^k:

$$x_\alpha = \xi^\alpha, \qquad x_3 = h\xi^3, \qquad \overset{0}{\boldsymbol{g}}_{\alpha} = \boldsymbol{e}_\alpha, \qquad \overset{0}{\boldsymbol{g}}_{3} = h\boldsymbol{e}_3, \qquad \overset{0}{J} = h, \tag{4.1}$$

where $\boldsymbol{e}_i$ is the orthonormal basis of the Cartesian coordinate system x_i.

We consider the stability problem of the plate with simply supported edges, which is compressed by a load of intensity p along the x_1 axis.

We use the following assumptions:

— the subcritical state is described by geometrically linear equations; therefore, we consider the linearized system (3.3)–(3.5);

— the subcritical stressed state is uniform:

$$\tau_{11} = -p, \qquad \tau_{13} = 0, \qquad \tau_{33} = 0; \tag{4.2}$$

— the plate material is isotropic:

$$\overset{0}{C}{}^{ijmn} = \lambda \overset{0}{g}{}^{ij} \overset{0}{g}{}^{mn} + \mu (\overset{0}{g}{}^{im} \overset{0}{g}{}^{jn} + \overset{0}{g}{}^{in} \overset{0}{g}{}^{jm}) \tag{4.3}$$

(λ and μ are the Lamé parameters) and by virtue of (4.1), we have

$$\overset{0}{g}{}^{\alpha\beta} = \delta^{\alpha\beta}, \qquad \overset{0}{g}{}^{\alpha 3} = 0, \qquad \overset{0}{g}{}^{33} = 1/h^2.$$

It follows from (4.2) and (4.3) that the only nonzero components $\tilde{C}^{ijmn}$ in (3.6) are

$$\begin{gathered} \tilde{C}^{1111} = \lambda + 2\mu - p, \qquad \tilde{C}^{1331} = (\mu - p)/h^2, \qquad \tilde{C}^{3333} = (\lambda + 2\mu)/h^4, \\ \tilde{C}^{1133} = \tilde{C}^{3311} = \tilde{C}^{1313} = \tilde{C}^{3131} = \tilde{C}^{3113} = \mu/h^2. \end{gathered} \tag{4.4}$$

For plane strain, we have

$$\Delta \boldsymbol{G}_2 = 0, \qquad \boldsymbol{e}_2 \cdot \Delta \boldsymbol{U}'' = 0. \tag{4.5}$$

We introduce the following notation:

$$\boldsymbol{e}_1 \cdot \Delta \boldsymbol{U}'' = u + \psi P_1 + [u]^2 P_2 + [u]^3 P_3, \qquad \boldsymbol{e}_3 \cdot \Delta \boldsymbol{U}'' = v + [v]^1 P_1 + [v]^2 P_2. \tag{4.6}$$

Using the properties of the Legendre polynomials and relations (4.1)–(4.6), from Hooke's law relations (3.4) we obtain

$$\begin{aligned} \boldsymbol{e}_1 \cdot \Delta \hat{\boldsymbol{T}}'^1 &= (\lambda + 2\mu - p)(u_{,1} P_0 + \psi_{,1} P_1) + \lambda([v]^1 P_0 + 3[v]^2 P_1), \\ \boldsymbol{e}_3 \cdot \Delta \hat{\boldsymbol{T}}''^1 &= (\mu - p) h v_{,1} + \mu([u]^1 + [u]^3), \\ \boldsymbol{e}_1 \cdot \Delta \hat{\boldsymbol{T}}^3 &= (\mu/h)(([u]^1 + [u]^3 + h v_{,1}) P_0 + 3[u]^2 P_1 + 5[u]^3 P_2), \\ \boldsymbol{e}_3 \cdot \Delta \hat{\boldsymbol{T}}^3 &= \lambda(u_{,1} P_0 + \psi_{,1} P_1) + ((\lambda + 2\mu)/h)([v]^1 P_0 + 3[v]^2 P_1). \end{aligned} \tag{4.7}$$

We call the unknown functions u, v, and ψ and their first derivatives that enter into (4.7) the main unknowns. The functions $[u]^2$, $[u]^3$, $[v]^1$, and $[v]^2$ are called the additional unknowns.

Since the faces $x_3 = \pm h$ are stress-free, the boundary conditions (3.5) become

$$\frac{\Delta \hat{\boldsymbol{T}}^3}{\overset{0}{J}} \bigg|_{\overset{0}{S}{}^{\pm}} = 0. \tag{4.8}$$

We substitute the last two relations in (4.7) into (4.8). As a result, we obtain a system of four linear algebraic equations for four additional unknown functions, whose solution has the form

$$[u]^2 = 0, \qquad [u]^3 = -\frac{1}{6}(\psi + h v_{,1}), \quad [v]^1 = -\frac{h\lambda}{3(\lambda + 2\mu)}\, u_{,1}, \quad [v]^2 = -\frac{h\lambda}{\lambda + 2\mu}\, \psi_{,1}. \tag{4.9}$$

Inserting (4.9) into (4.7) and the resulting expressions into (3.3), we arrive at the linear system of three ordinary differential equations for three main functions u, v, and ψ:

$$u_{,11} = 0, \quad \Big(\frac{(\lambda + 2\mu)^2 - \lambda^2}{\lambda + 2\mu} - p\Big)\psi_{,11} - \frac{5\mu}{2h^2}(\psi + h v_{,1}) = 0, \quad \psi_{,1} + \Big(1 - \frac{6p}{5\mu}\Big) h v_{,11} = 0. \tag{4.10}$$

The plate is simply supported at the edge surfaces $x = 0$ and $x = l$, which is equivalent to the following boundary conditions:

$$u = 0, \qquad v = 0, \qquad \psi_{,1} = 0 \qquad \text{for } x = 0, \; x = l. \tag{4.11}$$

Eliminating the function ψ from (4.10) and (4.11), we obtain the homogeneous linear ordinary fourth-order differential equation subject to homogeneous boundary conditions

$$\begin{gathered} \Big(\frac{(\lambda + 2\mu)^2 - \lambda^2}{\lambda + 2\mu} - p\Big)\Big(\frac{5\mu}{6} - p\Big) v_{,1111} + \frac{5\mu}{2h^2}\, p v_{,11} = 0, \\ v = 0, \qquad v_{,11} = 0 \qquad \text{for } x = 0, \; x = l. \end{gathered} \tag{4.12}$$

We seek the solution of problem (4.12) in the form

$$v = C \sin(m\pi x/l). \tag{4.13}$$

Substitution of (4.13) into (4.12) yields the equation for determining the critical load p

$$\Big(\frac{(\lambda + 2\mu)^2 - \lambda^2}{\lambda + 2\mu} - p\Big)\Big(\frac{5\mu}{6} - p\Big) m^2 \alpha^2 - \frac{5\mu}{2}\, p = 0, \tag{4.14}$$

where $\alpha = h\pi/l$.

For $m = 1$, the quadratic equation (4.14) takes the form

$$\begin{gathered} a p_*^2 - b p_* + c = 0, \\ a = \frac{4\alpha^4}{15(1 - \nu)}, \qquad b = \frac{4\alpha^2}{5(1 - \nu)} + \frac{\alpha^2}{3} + 1, \qquad c = 1, \end{gathered} \tag{4.15}$$

where ν is Poisson's ratio, $p_* = p/p_e$, $p_e = \alpha^2 E/(3(1-\nu^2))$ is the Euler critical load, and E is Young's modulus. With accuracy to α^4, the solution of Eq. (4.15) is written as

$$p_* \simeq 1 - \alpha^2\,(17 - 5\nu)/(15(1-\nu)). \tag{4.16}$$

The critical load obtained on the basis of the refined theory of plates has the form [5]:

$$p_1 \simeq 1 - \alpha^2 4/(5(1-\nu)). \tag{4.17}$$

The critical load determined in the three-dimensional linearized formulation with allowance for small subcritical strains was obtained in [6]:

$$p_2 \simeq 1 - \alpha^2\Big(\frac{2(6-\nu)}{15(1-\nu)} + \frac{1}{3}\Big). \tag{4.18}$$

Comparing the critical loads (4.16)–(4.18), we obtain the estimates

$$p_* \leqslant p_2 < p_1. \tag{4.19}$$

The equality sign in (4.19) is valid for $\nu = 0$.

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